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Second-Order Scalar-Tensor Field Equations in a Four-Dimensional Space

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Abstract

Lagrange scalar densities which are concomitants of a pseudo-Riemannian metric-tensor, a scalar field and their derivatives of arbitrary order are considered. The most general second-order Euler-Lagrange tensors derivable from such a Lagrangian in a four-dimensional space are constructed, and it is shown that these Euler-Lagrange tensors may be obtained from a Lagrangian which is at most of second order in the derivatives of the field functions.

1. *Introduction*

Our considerations will be based upon a real, four-dimensional, C^∞ differentiable manifold M. It will be assumed that all field functions are defined globally; however, our work will be of a purely local nature. By a pseudo-Riemannian metric for M we shall mean a C^∞ symmetric (0, 2) tensor field on M which associates a non-degenerate, symmetric bilinear form to each fibre of the tangent bundle of M. If x $(=x^i)$ is a chart for M the components of the metric will be denoted by g_{ij}, where Latin indices run from 1 to 4. The coefficients of the Christoffel connection determined by g_{ij} are †

$$\Gamma_j{}^i{}_k = \tfrac{1}{2}g^{ih}(g_{jh,k} + g_{kh,j} - g_{jk,h})$$

where g^{ih} is the matrix inverse of g_{ij} and an index k (say) preceded by a comma denotes a partial derivative with respect to the local coordinate x^k. If Y^i denotes the components of an arbitrary vector field of class C^2 then the components, $R_h{}^i{}_{jk}$, of the Riemann-Christoffel curvature tensor are defined by

$$Y^i{}_{|jk} - Y^i{}_{|kj} = Y^h R_h{}^i{}_{jk}$$

† The summation convention will be used throughout.

where a vertical bar preceding an index j (say) denotes a covariant derivative with respect to the Christoffel connection in the direction $\partial/\partial x^j$. Lastly, by way of preliminaries, the components of the Ricci tensor, scalar curvature and Einstein tensor are defined by

$$R_{hj} = R_h{}^i{}_{ji}, \qquad R = g^{hj}R_{hj} \qquad \text{and} \qquad G_{hj} = R_{hj} - \tfrac{1}{2}g_{hj}R$$

respectively, and $g = |\det(g_{ij})|$.

The vacuum field equations of most scalar-tensor field theories are usually assumed to be the Euler–Lagrange equations corresponding to some suitably chosen Lagrange scalar density which is a concomitant of a pseudo-Riemannian metric tensor. a scalar field and their derivatives (for example the Brans–Dicke (1961) field theory). Furthermore, it is usually demanded that the field equations be at most of second-order in the derivatives of both sets of field functions. Recently Horndeski & Lovelock (1972) have shown that in a four-dimensional space the most general second-order Euler–Lagrange equations which can be derived from a Lagrange scalar density of the form

$$L = L(g_{ij}; g_{ij,h}; g_{ij,hk}; \phi; \phi_{,i}) \tag{1.1}$$

(where ϕ is a scalar field) are given by†

$$\begin{aligned} E^{ab}(L) = {} & \surd(g)\beta_1{}'\delta^{acde}_{fhjk}g^{fb}\phi_{|c}{}^{|h}R_{de}{}^{jk} \\ & -\tfrac{1}{4}\surd(g)(\beta_3 + \tfrac{1}{2}\rho\beta_2)\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh} \\ & + \surd(g)(\beta_1{}'' + \tfrac{1}{4}\beta_2)\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}R_{de}{}^{jk} \\ & + \tfrac{1}{2}\surd(g)\beta_2\delta^{acd}_{efh}g^{eb}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} + \surd(g)\beta_3{}'\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e} \\ & + \tfrac{1}{2}\surd(g)\beta_2{}'\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} + \surd(g)(\rho\beta_3{}'' - \tfrac{1}{2}\eta)g^{ab} \\ & + \surd(g)(\dot{\eta} - \beta_3{}'')\phi^{|a}\phi^{|b} \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} E(L) = {} & -\surd(g)\beta_1{}'(R^2 - 4R_{ij}R^{ij} + R_{hijk}R^{hijk}) \\ & + \surd(g)(\beta_2{}'\phi_{|a}\phi_{|b} + 2\beta_2\phi_{|ab})G^{ab} - \surd(g)\beta_3{}'R \\ & + 2\surd(g)(\dot{\eta}'\rho + 2\ddot{\eta}\phi^{|a}\phi^{|b}\phi_{|ab} + \dot{\eta}g^{ab}\phi_{|ab} - \tfrac{1}{2}\eta') \end{aligned} \tag{1.3}$$

where β_1, β_2 and β_3 are arbitrary functions of ϕ, η is a function of ϕ and $\rho(=\phi_{,i}\phi_{,j}g^{ij})$, a prime denotes a partial derivative with respect to ϕ, a dot denotes a partial derivative with respect to ρ and for $h \geqslant 2$ the generalised Kronecker delta is defined by

$$\delta^{i_1 \ldots i_h}_{j_1 \ldots j_h} = \det \begin{vmatrix} \delta^{i_1}_{j_1} \ldots \delta^{i_1}_{j_h} \\ \vdots \quad\quad \vdots \\ \delta^{i_h}_{j_1} \ldots \delta^{i_h}_{j_h} \end{vmatrix}$$

† Indices will be 'lowered' and 'raised' by means of the metric and its inverse respectively.

Furthermore they established that the most general Lagrangian of the form (1.1) which yields (1.2) and (1.3) as its Euler-Lagrange tensors is given by

$$L = \tfrac{1}{4}\sqrt{(g)}\beta_1 \delta^{abcd}_{efhi} R_{ab}{}^{ef} R_{cd}{}^{hi} - \tfrac{1}{4}\sqrt{(g)}\beta_2 \delta^{abc}_{def} \phi_{|a}\phi^{|d} R_{bc}{}^{ef} + \tfrac{1}{2}\sqrt{(g)}\beta_3 \delta^{ab}_{cd} R_{ab}{}^{cd} + \sqrt{(g)}\eta + C\epsilon^{abcd} R^p{}_{qab} R^q{}_{pcd} \tag{1.4}$$

where ϵ^{abcd} is the four-dimensional Levi-Civita symbol and C is a constant.

However, the above result does *not* imply that (1.2) and (1.3) represent the most general second-order Euler-Lagrange tensors one could ever possibly obtain (in a four-dimensional space) from a variational principle in which the field functions are the components of a pseudo-Riemannian metric tensor and a scalar field. The purpose of this paper is to construct *the most general second-order Euler-Lagrange equations which can be obtained from a Lagrange scalar density of the form*

$$L = L(g_{ij}; g_{ij,i_1}; \ldots; g_{ij,i_1\ldots i_p}; \phi; \phi_{,i_1}; \ldots; \phi_{,i_1\ldots i_q}) \tag{1.5}$$

(where $p, q \geqslant 2$) *in a space of four-dimensions.* I shall now indicate the method which will be employed to construct these most general second-order Euler-Lagrange equations.

The Euler-Lagrange tensors associated with (1.5) are given by

$$E^{ij}(L) = \sum_{h=0}^{p} (-1)^{h+1} \frac{d}{dx^{i_1}} \cdots \frac{d}{dx^{i_h}} \frac{\partial L}{\partial g_{ij,i_1\ldots i_h}} \tag{1.6}$$

and

$$E(L) = \sum_{h=0}^{q} (-1)^{h+1} \frac{d}{dx^{i_1}} \cdots \frac{d}{dx^{i_h}} \frac{\partial L}{\partial \phi_{,i_1\ldots i_h}} \tag{1.7}$$

where (1.6) is obtained from (1.5) through a variation of the g_{ij}'s regarding ϕ (and its derivatives) as an arbitrary preassigned function of position, and (1.7) is obtained from (1.5) through a variation of ϕ regarding the g_{ij}'s (and their derivatives) as arbitrary preassigned functions of position. It should be noted that in general $E^{ij}(L)$ is of $2p$th order in the derivatives of g_{ij} and of $(p+q)$th order in the derivatives of ϕ; whereas $E(L)$ is of $2q$th order in the derivatives of ϕ and of $(p+q)$th order in the derivatives of g_{ij}.

Using techniques similar to those employed by du Plessis (1969) it can be shown (see Horndeski (1973)) that $E^{ij}(L)$ and $E(L)$ are related by

$$E^{ij}(L)_{|j} = \tfrac{1}{2}\phi^{|i}E(L) \tag{1.8}$$

This result generalises a similar identity (Horndeski & Lovelock (1972)), and shows that the Euler-Lagrange equation, $E(L) = 0$, is a consequence of the Euler-Lagrange equations $E^{ij}(L) = 0$.

We shall now demand that both $E^{ij}(L)$ and $E(L)$ be at most of *second-order* in the derivatives of both g_{ij} and ϕ. In general if $E^{ij}(L)$ is of second-order then $E^{ij}(L)_{|j}$ will be of third order, however, due to (1.8), we see that under the above restrictions $E^{ij}(L)_{|j}$ must be of second-order. This places a severe constraint upon $E^{ij}(L)$ and leads us to consider the following problem: In a space of dimension 4 find the most general symmetric tensor density of the form

$$A^{ij} = A^{ij}(g_{ab}; g_{ab,c}; g_{ab,cd}; \phi; \phi_{,c}; \phi_{,cd}) \tag{1.9}$$

which is such that

$$A^{ij}{}_{|j} = A^{ij}{}_{|j}(g_{ab}; g_{ab,c}; g_{ab,cd}; \phi; \phi_{,c}; \phi_{,cd}) \tag{1.10}$$

In the light of the above observations it is clear that the E^{ij} we seek will be contained in the above A^{ij}. Section 2 is devoted to constructing the most general A^{ij} which satisfies (1.9) and (1.10) in a space of four-dimensions. The A^{ij} so constructed will be shown to involve ten arbitrary functions of ϕ and ρ.

Now if A^{ij} were in fact the Euler–Lagrange tensor (i.e., the E^{ij}) of some Lagrangian of the form (1.5) then due to (1.8) we should be able to express $A^{ij}{}_{|j}$ as follows

$$A^{ij}{}_{|j} = \phi^{|i} A \tag{1.11}$$

where A is a scalar density of the form

$$A = A(g_{ab}; g_{ab,c}; g_{ab,cd}; \phi; \phi_{,c}; \phi_{,cd}) \tag{1.12}$$

However, the symmetric tensor density constructed in Section 2 does *not* satisfy (1.11) identically. Nevertheless, when we demand that (1.11) be satisfied we obtain a system of six first-order partial differential equations relating the ten arbitrary functions of ϕ and ρ which appear in A^{ij}. The paper concludes by showing that when the ten arbitrary functions appearing in A^{ij} are so related A^{ij} is indeed the Euler–Lagrange tensor of a suitably chosen Lagrange scalar density of the form (1.5) with $p = q = 2$. This in turn resolves the original problem of determining the most general second-order Euler–Lagrange tensors which may be obtained from a Lagrangian of the form (1.5) in a space of four dimensions.

Remark: It should be noted that the approach taken in the consideration of the above problem is quite similar to the one used by Lovelock (1970b, 1971) where he constructs all tensor densities $\mathscr{A}^{ij}$ of the form

$$\mathscr{A}^{ij} = \mathscr{A}^{ij}(g_{hk}; g_{hk,c}; g_{hk,cd})$$

which are such that

$$\mathscr{A}^{ij}{}_{|j} = 0$$

Lovelock has also shown that all such $\mathscr{A}^{ij}$ are the Euler–Lagrange tensors corresponding to a Lagrange scalar density of the form

$$L = L(g_{hk}; g_{hk,c}; g_{kh,cd})$$

2. *The Construction of Certain Symmetric Tensor Densities*

In this section we wish to construct (in a four-dimensional space) all symmetric contravariant tensor densities of rank 2 which have their components, A^{ij}, satisfying the following conditions:

(i) A^{ij} is of the form

$$A^{ij} = A^{ij}(g_{hk}; g_{hk,c}; g_{hk,cd}; \phi; \phi_{,c}; \phi_{,cd}) \tag{2.1}$$

(ii) The covariant divergence of A^{ij}, viz., $A^{ij}{}_{|j}$, is at most of second-order in the derivatives of both g_{ij} and ϕ.

It will be assumed that A^{ij}, and all other functions which arise in this paper, have as many continuous derivatives as we wish.

Due to equation (2.1) we see that condition (ii) is equivalent to

$$\frac{\partial A^{ij}{}_{|j}}{\partial g_{rs,tvu}} = 0 \tag{2.2}$$

and

$$\frac{\partial A^{ij}{}_{|j}}{\partial \phi_{,rst}} = 0 \tag{2.3}$$

In order to simplify the form of the ensuing expressions we adopt the following notation: If

$$B\text{:::} = B\text{:::}(g_{hk}; g_{hk,c}; g_{hk,cd}; \phi; \phi_{,c}; \phi_{,cd})$$

is any quantity then we define

$$\left.\begin{aligned} B\text{:::}^{;ab,cd} &= \frac{\partial B\text{:::}}{\partial g_{ab,cd}} \\ \text{and} \qquad & \\ B\text{:::}^{;ab} &= \frac{\partial B\text{:::}}{\partial \phi_{,ab}} \end{aligned}\right\} \tag{2.4a}$$

so that, for example,

$$\left.\begin{aligned} A^{ij;ab,cd} &= \frac{\partial A^{ij}}{\partial g_{ab,cd}}, \qquad A^{ij;ab} = \frac{\partial A^{ij}}{\partial \phi_{,ab}} \\ \text{and} \qquad & \\ A^{ij;ab;cd,ef;rs} &= \frac{\partial}{\partial \phi_{,rs}} \frac{\partial}{\partial g_{cd,ef}} \frac{\partial}{\partial \phi_{,ab}} A^{ij} \end{aligned}\right\} \tag{2.4b}$$

Since A^{ij} is a tensor density of the form (2.1) it is easily seen that repeated partial differentiation of A^{ij} with respect to $g_{ab,cd}$ and $\phi_{,ab}$ will yield tensor densities. Consequently the quantities presented in equation (2.4b) are tensorial. Furthermore, since $A^{ij}{}_{|j}$ is (in general) a concomitant of g_{ij} and its first three derivatives, along with ϕ and its first three derivatives, $\partial A^{ij}{}_{|j}/\partial g_{rs,tvu}$ and $\partial A^{ij}{}_{|j}/\partial \phi_{,rst}$ are tensor densities. Thus equations (2.2) and (2.3) are tensorial conditions.

Since A^{ij} is a tensor density of the form (2.1) $A^{ij;ab,cd}$ must satisfy the 'invariance identity' (see, e.g. Rund (1964, 1966), du Plessis (1969) or Horndeski & Lovelock (1972)):

$$A^{ij;ab,cd} + A^{ij;ac,db} + A^{ij;ad,bc} = 0 \tag{2.5}$$

Through repeated use of (2.5) in conjunction with the fact that

$$A^{ij;ab,cd} = A^{ij;ba,cd} = A^{ij;ab,dc} \tag{2.6}$$

it is readily shown that

$$A^{ij;ab,cd} = A^{ij;cd,ab} \tag{2.7}$$

We shall now turn our attention to equations (2.2) and (2.3).

Due to the fact that A^{ij} is a tensor density of the form (2.1) we have

$$A^{ij}{}_{|j} = \frac{\partial A^{ij}}{\partial g_{ab}} g_{ab,j} + \frac{\partial A^{ij}}{\partial g_{ab,c}} g_{ab,cj} + A^{ij;ab,cd} g_{ab,cdj}$$
$$+ \frac{\partial A^{ij}}{\partial \phi} \phi_{,j} + \frac{\partial A^{ij}}{\partial \phi_{,a}} \phi_{,aj} + A^{ij;ab} \phi_{,abj} + A^{kj}\Gamma_k{}^i{}_j \tag{2.8}$$

Thus we see that (2.2) and (2.3) will hold if and only if

$$A^{ij;ab,cd} \frac{\partial g_{ab,cdj}}{\partial g_{rs,tvu}} = 0 \tag{2.9}$$

and

$$A^{ij;ab} \frac{\partial \phi_{,abj}}{\partial \phi_{,rst}} = 0 \tag{2.10}$$

respectively. Upon performing the indicated differentiations we find that (2.9) and (2.10) reduce to

$$A^{iu;rs,tv} + A^{it;rs,vu} + A^{iv;rs,ut} = 0 \tag{2.11}$$

and

$$A^{it;rs} + A^{ir;st} + A^{is;tr} = 0 \tag{2.12}$$

respectively.

Thus we have established

Theorem 2.1. *A symmetric contravariant tensor density of the form*

$$A^{ij} = A^{ij}(g_{hk}; g_{hk,c}; g_{hk,cd}; \phi; \phi_{,c}; \phi_{,cd}) \tag{2.1}$$

will have its covariant divergence, $A^{ij}{}_{|j}$, being at most of second-order in the derivatives of both g_{ij} and *ϕ if and only if*

$$A^{iu;rs,tv} + A^{it;rs,vu} + A^{iv;rs,ut} = 0 \tag{2.11}$$

and

$$A^{it;rs} + A^{ir;st} + A^{is;tr} = 0 \tag{2.12}$$

Due to our previous remarks it should be clear that equations (2.11) and (2.12) are tensorial equations.

Through repeated use of equations (2.11) and (2.12) in conjunction with equations (2.6) and the symmetries of $A^{ij;ab,cd}$ and $A^{ij;ab}$ we may conclude that whenever A^{ij} satisfies conditions (i) and (ii) (see the first paragraph of this section) then

$$A^{ij;ab,cd} = A^{cd;ab,ij} = A^{ab;ij,cd} \tag{2.13}$$

and

$$A^{ij;ab} = A^{ab;ij} \tag{2.14}$$

In order to proceed with our study of conditions (i) and (ii) it will be convenient for us to introduce the following definition which is used by Lovelock (1970b): A quantity $B^{i_1 i_2 \cdots i_{2h-1} i_{2h} \cdots i_{2p}}$ $(p > 1)$ is said to enjoy property S if it satisfies the following three conditions:

(A) it is symmetric in $i_{2h-1} i_{2h}$ for $h = 1, \ldots, p$;
(B) it is symmetric under the interchange of the pair $(i_1 i_2)$ with the pair $(i_{2h-1} i_{2h})$ for $h = 2, \ldots, p$;
(C) it satisfies the cyclic identity involving any three of the four indices $(i_1 i_2)(i_{2h-1} i_{2h})$ for $h = 2, \ldots, p$; e.g., when $h = 2$

$$B^{i_1 i_2 i_3 i_4 \cdots i_{2p}} + B^{i_2 i_3 i_1 i_4 \cdots i_{2p}} + B^{i_3 i_1 i_2 i_4 \cdots i_{2p}} = 0$$

A quantity B^{ab} is said to have property S if $B^{ab} = B^{ba}$.

Using equations (2.5)-(2.7) and (2.11)-(2.14) we may conclude that

$$A^{ab;i_1 i_2, i_3 i_4; \ldots ; i_{4h-3} i_{4h-2}, i_{4h-1} i_{4h}; i_{4h+1} i_{4h+2}; \ldots ; i_{4h+2k-1} i_{4h+2k}} \tag{2.15}$$

enjoys property S when the non-negative integers h and k are such that $h + k \geqslant 0$.

The following result has been established by Lovelock (1970b):

Lemma 2.1. *If $B^{i_1 \cdots i_{4M+2}}$ is any quantity which has property S then it vanishes whenever three (or more) indices are equal. In particular $B^{i_1 \cdots i_{4M+2}}$ vanishes identically (in a four-dimensional space) if $M \geqslant 2$.*

Upon replacing $B^{i_1 \cdots i_{4M+2}}$, in the above lemma, by the tensor density introduced in (2.15) we may conclude that

Corollary 2.1. *Whenever k is an even non-negative integer and $h + \frac{1}{2}k \geqslant 2$ then*

$$A^{ab;i_1 i_2, i_3 i_4; \ldots ; i_{4h-3} i_{4h-2}, i_{4h-1} i_{4h}; i_{4h+1} i_{4h+2}; \ldots ; i_{4h+2k-1} i_{4h+2k}} = 0 \tag{2.16}$$

Equation (2.16) will permit us to construct the most general symmetric contravariant tensor density of rank 2 which satisfies conditions (i) and (ii) in a space of four-dimensions.

To begin with when $h = 2$ and $k = 0$ we may use (2.16) to conclude that

$$A^{ab;i_1i_2,i_3i_4;i_5i_6,i_7i_8} = 0$$

Upon integrating this expression twice with respect to $g_{cd,ef}$ we find

$$A^{ab} = \beta^{abcdef} g_{cd,ef} + \beta^{ab} \tag{2.17}$$

where β^{ab} and β^{abcdef} are concomitants of $g_{ij}, g_{ij,h}, \phi, \phi_{,i}$ and $\phi_{,ij}$ which enjoy property S.

Now it is easily seen that

$$\beta^{abcdef} g_{cd,ef} = \tfrac{2}{3}\beta^{abcdef} R_{ecdf} + J^{ab}$$

where J^{ab} is symmetric in a and b and is a concomitant of $g_{ij}, g_{ij,h}, \phi, \phi_{,i}$ and $\phi_{,ij}$. Consequently we can write (2.17) as follows:

$$A^{ab} = \hat{\beta}^{abcdef} R_{ecdf} + \hat{\beta}^{ab} \tag{2.18}$$

where $\hat{\beta}^{ab}$ and $\hat{\beta}^{abcdef}$ are functions of $g_{ij}, g_{ij,h}, \phi, \phi_{,i}$ and $\phi_{,ij}$ which possess property S. Clearly $\hat{\beta}^{ab}$ and $\hat{\beta}^{abcdef}$ must be tensor densities.

We shall now show how to construct the $\hat{\beta}^{ab\cdots}$ in terms of $\phi_{|ab}$ and tensor densities which enjoy property S and are concomitants of g_{ij}, ϕ and $\phi_{,i}$.

To begin with we may combine Corollary 2.1 (with $k = 2$ and $h = 1$) together with equation (2.18) to conclude that

$$\hat{\beta}^{abcdef;hi} = \hat{\beta}^{abcdef;hi}(g_{ij}; g_{ij,h}; \phi; \phi_{,i}) \tag{2.19}$$

In order to proceed further it will be necessary to make use of the following result (the proof of which may be found in Horndeski (1971)):

Lemma 2.2. *If*

$$T^{i_1\cdots i_r}_{j_1\cdots j_s} = T^{i_1\cdots i_r}_{j_1\cdots j_s}(g_{ij}; g_{ij,h}; g_{ij,hk}; \phi; \phi_{,i})$$

is a relative tensor density of class C^1 which is of contravariant valence r and covariant valence s and furthermore is such that

$$T^{i_1\cdots i_r;ab,cd}_{j_1\cdots j_s} \equiv 0$$

then

$$T^{i_1\cdots i_r}_{j_1\cdots j_s} = T^{i_1\cdots i_r}_{j_1\cdots j_s}(g_{ij}; \phi; \phi_{,i})$$

Due to Lemma 2.2 equation (2.19) becomes

$$\hat{\beta}^{abcdef;hi} = \hat{\beta}^{abcdef;hi}(g_{ij}; \phi; \phi_{,i})$$

and thus

$$\hat{\beta}^{abcdef} = \xi^{abcdefhi}\phi_{,hi} + \alpha^{abcdef}$$

where $\xi^{abcdefhi}$ is an arbitrary tensor density with property S and a concomitant of g_{ij}, ϕ and $\phi_{,i}$; and the α^{abcdef} are arbitrary functions of g_{ij}, $g_{ij,h}$, ϕ and $\phi_{,i}$ which enjoy property S. If in the above expression $\phi_{,hi}$ is replaced by $\phi_{|hi} + \phi_{|r}\Gamma_h{}^r{}_i$ then the tensorial properties of $\hat{\beta}^{abcdef}$, $\xi^{abcdefhi}$ and $\phi_{|hi}$ may be used together with Lemma 2.2 to conclude that

$$\hat{\beta}^{abcdef} = \xi^{abcdefhi}\phi_{|hi} + \xi^{abcdef} \tag{2.20}$$

where ξ^{abcdef} is an arbitrary tensor density with property S and a concomitant of g_{ij}, ϕ and $\phi_{,i}$.

Similarly equations (2.18) and (2.20) may be employed in conjunction with Corollary 2.1 (when $h = 0$ and $k = 4$) to show that

$$\hat{\beta}^{ab} = \psi^{abcdefhi}\phi_{|cd}\phi_{|ef}\phi_{|hi} + \psi^{abcdef}\phi_{|cd}\phi_{|ef} + \psi^{abcd}\phi_{|cd} + \psi^{ab} \tag{2.21}$$

where each $\psi^{ab\cdots}$ is an arbitrary tensor density with property S and a concomitant of g_{ij}, ϕ and $\phi_{,i}$.

Combining equations (2.18), (2.20) and (2.21) we find:

$$\begin{aligned} A^{ab} = {} & \xi^{abcdefhi}R_{ecdf}\phi_{|hi} + \xi^{abcdef}R_{ecdf} \\ & + \psi^{abcdefhi}\phi_{|cd}\phi_{|ef}\phi_{|hi} + \psi^{abcdef}\phi_{|cd}\phi_{|ef} \\ & + \psi^{abcd}\phi_{|cd} + \psi^{ab} \end{aligned} \tag{2.22}$$

Employing techniques similar to those used in Lovelock (1969, 1970b, 1971) and Horndeski & Lovelock (1972) it can be shown that the most general tensor densities of the form

$$\theta^{i_1\cdots i_{2k}} = \theta^{i_1\cdots i_{2k}}(g_{ij}; \phi; \phi_{,i})$$

($k = 1, \ldots, 4$) which enjoy property S in a space of four-dimensions are given by:

$$\theta^{i_1 i_2} = \sqrt{(g)}C_1 g^{i_1 i_2} + \sqrt{(g)}C_2\phi^{,i_1}\phi^{,i_2} \tag{2.23}$$

$$\begin{aligned} \theta^{i_1 i_2 i_3 i_4} = {} & \sqrt{(g)}C_3(g^{i_1 i_3}g^{i_2 i_4} + g^{i_1 i_4}g^{i_2 i_3} - 2g^{i_1 i_2}g^{i_3 i_4}) \\ & + \sqrt{(g)}C_4(\phi^{,i_1}\phi^{,i_3}g^{i_2 i_4} + \phi^{,i_2}\phi^{,i_4}g^{i_1 i_3} + \phi^{,i_1}\phi^{,i_4}g^{i_2 i_3} \\ & + \phi^{,i_2}\phi^{,i_3}g^{i_1 i_4} - 2[\phi^{,i_1}\phi^{,i_2}g^{i_3 i_4} + \phi^{,i_3}\phi^{,i_4}g^{i_1 i_2}]) \end{aligned} \tag{2.24}$$

$$\begin{aligned} \theta^{i_1\cdots i_6} = {} & \frac{1}{\sqrt{(g)}}(C_5\phi_{,r}\phi_{,s} + C_6 g_{rs})\{\epsilon^{i_1 i_3 i_5 r}\epsilon^{i_2 i_4 i_6 s} \\ & + \epsilon^{i_1 i_3 i_6 r}\epsilon^{i_2 i_4 i_5 s} + \epsilon^{i_1 i_4 i_5 r}\epsilon^{i_2 i_3 i_6 s} + \epsilon^{i_1 i_4 i_6 r}\epsilon^{i_2 i_3 i_5 s}\} \end{aligned} \tag{2.25}$$

and

$$\theta^{i_1\cdots i_8} = \frac{C_7}{\sqrt{(g)}}\{\epsilon^{i_1i_3i_5i_7}\epsilon^{i_2i_4i_6i_8} + \epsilon^{i_1i_3i_5i_8}\epsilon^{i_2i_4i_6i_7}$$
$$+ \epsilon^{i_1i_3i_6i_7}\epsilon^{i_2i_4i_5i_8} + \epsilon^{i_1i_3i_6i_8}\epsilon^{i_2i_4i_5i_7} + \epsilon^{i_2i_3i_5i_7}\epsilon^{i_1i_4i_6i_8}$$
$$+ \epsilon^{i_2i_3i_5i_8}\epsilon^{i_1i_4i_6i_7} + \epsilon^{i_2i_3i_6i_7}\epsilon^{i_1i_4i_5i_8} + \epsilon^{i_2i_3i_6i_8}\epsilon^{i_1i_4i_5i_7}\} \tag{2.26}$$

where $C_1, \ldots, C_7$ are arbitrary functions of ϕ and $\rho(=\phi_{,i}\phi_{,j}g^{ij})$.

Making use of equations (2.23)-(2.26) together with the symmetry properties of the Riemann-Christoffel curvature tensor we find, after much simplification, that (2.22) becomes

$$A^{ab} = \sqrt{(g)}\{K_1\delta^{acde}_{fhjk}g^{fb}\phi_{|c}{}^{|h}R_{de}{}^{jk} + K_2\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh}$$
$$+ K_3\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}R_{de}{}^{jk} + K_4\delta^{acde}_{fhjk}g^{fb}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}$$
$$+ K_5\delta^{acd}_{efh}g^{eb}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} + K_6\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}$$
$$+ K_7\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e} + K_8\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} + K_9g^{ab}$$
$$+ K_{10}\phi^{|a}\phi^{|b}\} \tag{2.27}$$

where $K_1, \ldots, K_{10}$ are arbitrary differentiable functions of ϕ and ρ.

Thus we have established

Theorem 2.2. *In a space of dimension four the most general symmetric contravariant tensor density of the form*

$$A^{ab} = A^{ab}(g_{ij}; g_{ij,h}; g_{ij,hk}; \phi; \phi_{,h}; \phi_{,hk})$$

which is such that $A^{ab}{}_{|b}$ is at most of second-order in the derivatives of both g_{ij} and ϕ is given by equation (2.27).

3. *The Consequences of Demanding that $A^{ab}{}_{|b} = \phi^{|a}A$*

In the introduction it was pointed out that if the contravariant components, A^{ab}, of a symmetric tensor density are of the form

$$A^{ab} = A^{ab}(g_{ij}; g_{ij,h}; g_{ij,hk}; \phi; \phi_{,h}; \phi_{,hk}) \tag{3.1}$$

and furthermore are such that $A^{ab}{}_{|b}$ is at most of second-order in the derivatives of both g_{ij} and ϕ then a necessary condition for A^{ab} to be the Euler-Lagrange tensor of some Lagrange scalar density of the form (1.5) is that there exists a scalar density A for which

$$A^{ab}{}_{|b} = \phi^{|a}A \tag{3.2}$$

with

$$A = A(g_{ij}; g_{ij,h}; g_{ij,hk}; \phi; \phi_{,h}; \phi_{,hk})$$

Obviously those contravariant symmetric tensor densities of rank 2 which have their components satisfying (3.1) and (3.2) are contained in the class of symmetric tensor densities of rank 2 with components, A^{ab}, being of the form (3.1) and such that $A^{ab}{}_{|b}$ is at most of second-order in the derivatives of both g_{ij} and ϕ. At the end of the preceding section the most general element of the latter class of tensor densities was presented for spaces of four-dimensions. We shall now proceed to examine the divergence of this tensor and as a consequence determine the necessary and sufficient conditions under which it is of the form (3.2).

Using equation (2.27) in conjunction with the Ricci and Bianchi identities we find, after rearrangement, that

$$\begin{aligned} A^{ab}{}_{|b} = {} & \sqrt{(g)}\{K'_1\delta^{acde}_{fhjk}\phi^{|f}\phi_{|c}{}^{|h}R_{de}{}^{jk} + 2\dot{K}_2\delta^{acd}_{efh}\phi_{|p}\phi^{|pe}R_{cd}{}^{fh} \\ & + K_3\delta^{acde}_{fhjk}\phi^{|h}\phi_{|c}{}^{|f}R_{de}{}^{jk} + K_5\delta^{acd}_{efh}\phi^{|m}R_{mc}{}^{fe}\phi_{|d}{}^{|h}\} \\ & + \sqrt{(g)}\{2\dot{K}_1\delta^{acde}_{fhjk}\phi_{|p}\phi^{|pf}\phi_{|c}{}^{|h}R_{de}{}^{jk} \\ & + \tfrac{3}{2}K_4\delta^{acde}_{fhjk}\phi^{|m}R_{mc}{}^{hf}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}\} \\ & + \tfrac{1}{2}\sqrt{(g)}K_1\delta^{acde}_{fhjk}\phi^{|m}R_{mc}{}^{hf}R_{de}{}^{jk} + \sqrt{(g)}\{K'_2\delta^{acd}_{efh}\phi^{|e}R_{cd}{}^{fh} \\ & + \tfrac{1}{2}K_7\delta^{ac}_{de}\phi^{|m}R_{mc}{}^{ed}\} + \tfrac{1}{2}\sqrt{(g)}K_8\delta^{acd}_{efh}\phi_{|c}\phi^{|f}\phi^{|m}R_{md}{}^{he} \\ & + \sqrt{(g)}\{2\dot{K}_3\delta^{acde}_{fhjk}\phi_{|p}\phi^{|pf}\phi_{|c}\phi^{|h}R_{de}{}^{jk} \\ & + K_6\delta^{acde}_{fhjk}\phi_{|c}\phi^{|h}\phi^{|m}R_{md}{}^{jf}\phi_{|e}{}^{|k}\} + \sqrt{(g)}\{K'_4\delta^{acde}_{fhjk}\phi^{|f}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\ & + 2\dot{K}_5\delta^{acd}_{efh}\phi_{|p}\phi^{|pe}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} \\ & + K_6\delta^{acde}_{fhjk}\phi^{|h}\phi_{|c}{}^{|f}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}\} \\ & + 2\sqrt{(g)}\dot{K}_8\delta^{acd}_{efh}\phi_{|p}\phi^{|pe}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} \\ & + 2\sqrt{(g)}\dot{K}_6\delta^{acde}_{fhjk}\phi_{|p}\phi^{|pf}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\ & + 2\sqrt{(g)}\dot{K}_4\delta^{acde}_{fhjk}\phi_{|p}\phi^{|pf}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\ & + \sqrt{(g)}\{K'_5\delta^{acd}_{efh}\phi^{|e}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} + 2\dot{K}_7\delta^{ac}_{de}\phi_{|p}\phi^{|pd}\phi_{|c}{}^{|e} \\ & + K_8\delta^{acd}_{efh}\phi^{|f}\phi_{|c}{}^{|e}\phi_{|d}{}^{|h}\} + \sqrt{(g)}\{(2\dot{K}_9 + K_{10})\phi_{|b}\phi^{|ab} \\ & + K'_7\delta^{ac}_{de}\phi^{|d}\phi_{|c}{}^{|e}\} + \sqrt{(g)}\phi^{|a}\{K'_9 + \rho K'_{10} + 2\dot{K}_{10}\phi^{|b}\phi^{|c}\phi_{|bc} \\ & + K_{10}\phi_{|c}{}^{|c}\} \end{aligned} \tag{3.3}$$

where a prime denotes a partial derivative with respect to ϕ and a dot denotes a partial derivative with respect to ρ.

Using the fact that in a space of four-dimensions†

$$\delta^{abcde}_{fhjkm}\phi^{|f}R_{bc}{}^{hj}R_{de}{}^{km} = 0 \tag{3.4}$$

$$\delta^{abcde}_{fhjkm}\phi^{|f}\phi_{|b}{}^{|h}\phi_{|c}{}^{|j}R_{de}{}^{km} = 0 \tag{3.5}$$

† This type of approach to the derivation of dimensionally dependent identities is due to Lovelock (1967, 1970a).

and

$$\delta^{abcde}_{fhjkm}\phi^{|f}\phi_{|b}{}^{|h}\phi_{|c}{}^{|j}\phi_{|d}{}^{|k}\phi_{|e}{}^{|m} = 0 \tag{3.6}$$

together with the symmetry properties of the generalised Kronecker delta it can be shown that (3.3) may be rewritten as follows:

$$\begin{aligned} A^{ab}{}_{|b} = {} & \sqrt{(g)}\phi^{|a}Q + \sqrt{(g)}\{\alpha\,\delta^{ace}_{hjk}\phi^{|d}\phi_{|c}{}^{|h}R_{de}{}^{jk} \\ & + \beta\,\delta^{ade}_{hjk}\phi^{|c}\phi_{|c}{}^{|h}R_{de}{}^{jk} - \gamma\,\delta^{ace}_{fjk}\phi_{|p}\phi^{|pf}\phi_{|c}\phi^{|d}R_{de}{}^{jk} \\ & + \epsilon\,\delta^{abce}_{hjkm}\phi^{|p}R_{pb}{}^{jh}\phi_{|c}{}^{|k}\phi_{|e}{}^{|m} + \mu\,\delta^{ac}_{de}\phi^{|m}R_{mc}{}^{ed} \\ & + \nu\,\delta^{ade}_{fjk}\phi^{|p}\phi_{|p}{}^{|f}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} + 2\omega\,\delta^{ac}_{de}\phi^{|p}\phi_{|p}{}^{|d}\phi_{|c}{}^{|e} \\ & + \xi\,\phi_{|p}\phi^{|ap}\} \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} Q = {} & \dot{K}_1\delta^{bcde}_{hjkm}\phi_{|b}{}^{|h}\phi_{|c}{}^{|j}R_{de}{}^{km} \\ & + (K_1' - K_3)\delta^{cde}_{fjk}\phi_{|c}{}^{|f}R_{de}{}^{jk} - 2\dot{K}_3\delta^{cde}_{fjk}\phi^{|p}\phi_{|p}{}^{|f}\phi_{|c}R_{de}{}^{jk} \\ & - K_6\delta^{cde}_{fjk}\phi_{|c}\phi^{|m}R_{md}{}^{jf}\phi_{|e}{}^{|k} - \tfrac{1}{8}K_1\delta^{bcde}_{hjkm}R_{bc}{}^{hj}R_{de}{}^{km} \\ & + K_2'\delta^{cd}_{fh}R_{cd}{}^{fh} - \tfrac{1}{2}K_8\delta^{cd}_{eh}\phi_{|c}\phi^{|m}R_{md}{}^{he} \\ & - 2\dot{K}_8\delta^{cd}_{eh}\phi_{|p}\phi^{|pe}\phi_{|c}\phi_{|d}{}^{|h} + \tfrac{1}{2}\dot{K}_4\delta^{bcde}_{hjkm}\phi_{|b}{}^{|h}\phi_{|c}{}^{|j}\phi_{|d}{}^{|k}\phi_{|e}{}^{|m} \\ & - 2\dot{K}_6\delta^{cde}_{fjk}\phi^{|p}\phi_{|p}{}^{|f}\phi_{|c}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\ & + (K_4' - K_6)\delta^{cde}_{hjk}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} + (K_5' - K_8)\delta^{cd}_{fh}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} \\ & + K_9' + \rho K_{10}' + 2\dot{K}_{10}\phi^{|b}\phi^{|c}\phi_{|bc} + (K_{10} + K_7')\phi_{|c}{}^{|c} \end{aligned} \tag{3.8}$$

$$\left.\begin{aligned} & \alpha = 2K_1' - 2K_3 + K_5 + \rho K_6; \qquad \beta = 2\dot{K}_2 - K_1' + K_3 + 2\rho\dot{K}_3 \\ & \gamma = 4\dot{K}_3 + K_6; \qquad \epsilon = 2\dot{K}_1 + \tfrac{3}{2}K_4; \qquad \mu = 2K_2' + \tfrac{1}{2}K_7 + \tfrac{1}{2}\rho K_8 \\ & \nu = 2\dot{K}_5 + 3K_6 - 3K_4' + 2\rho\dot{K}_6; \qquad \omega = \dot{K}_7 - K_5' + K_8 + \rho\dot{K}_8 \\ & \text{and} \\ & \xi = 2\dot{K}_9 + K_{10} - K_7' \end{aligned}\right\} \tag{3.9}$$

In the light of equation (3.7) it is clear that $A^{ab}{}_{|b}$ will satisfy (3.2) if and only if there exists a scalar density B of the form

$$B = B(g_{ij}; g_{ij,h}; g_{ij,hk}; \phi; \phi_{,i}; \phi_{,ij}) \tag{3.10}$$

which is such that

$$\begin{aligned} \phi^{|a}B = {} & \sqrt{(g)}\{\alpha\,\delta^{ace}_{hjk}\phi^{|d}\phi_{|c}{}^{|h}R_{de}{}^{jk} + \beta\,\delta^{ade}_{hjk}\phi^{|c}\phi_{|c}{}^{|h}R_{de}{}^{jk} \\ & - \gamma\,\delta^{ace}_{fjk}\phi_{|p}\phi^{|pf}\phi_{|c}\phi^{|d}R_{de}{}^{jk} + \epsilon\,\delta^{abce}_{hjkm}\phi^{|p}R_{pb}{}^{jh}\phi_{|c}{}^{|k}\phi_{|e}{}^{|m} \\ & + \mu\,\delta^{ac}_{de}\phi^{|m}R_{mc}{}^{ed} + \nu\,\delta^{ade}_{fjk}\phi^{|p}\phi_{|p}{}^{|f}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\ & + 2\omega\,\delta^{ac}_{de}\phi^{|p}\phi_{|p}{}^{|d}\phi_{|c}{}^{|e} + \xi\,\phi_{|p}\phi^{|ap}\} \end{aligned} \tag{3.11}$$

Equation (3.11) must hold at every point of our manifold M (since the field functions were assumed to be globally defined and we are working on the domain of an arbitrary coordinate chart) and for every pseudo–Riemannian metric and scalar field defined on M. Furthermore, we are regarding (3.11) as an equation to be solved for B. It will now be shown that (3.11) admits a solution if and only if

$$\alpha = \beta = \gamma = \epsilon = \mu = \nu = \omega = \xi = 0 \tag{3.12}$$

and that that solution is

$$B = 0 \tag{3.13}$$

Clearly if (3.12) holds then the solution to (3.11) is $B = 0$. Thus we need only show that if (3.11) admits a solution, B, then (3.12) must hold, and consequently B must vanish.

In order to establish this result we begin by differentiating (3.11) once with respect to $g_{rs,tv}$ and twice with respect to $\phi_{,uw}$ to obtain

$$\phi^{|a}B^{;iq;uw;rs,tv} = \epsilon\sqrt{(g)}\delta^{abce}_{hjkm}\phi^{|p}\frac{\partial^2(\phi_{|c}{}^{|k}\phi_{|e}{}^{|m})}{\partial\phi_{,iq}\partial\phi_{,uw}}g^{jd}g^{hf}\frac{\partial R_{pbdf}}{\partial g_{rs,tv}} \tag{3.14}$$

It is easily shown that

$$\frac{\partial R_{abcd}}{\partial g_{hi,jk}} = \tfrac{1}{4}\{D^{hijk}_{abcd} + D^{hijk}_{cadb} - D^{hijk}_{abdc} - D^{hijk}_{dacb}\} \tag{3.15}$$

where

$$D^{hijk}_{abcd} = \tfrac{1}{2}(\delta^h_a\delta^i_d + \delta^h_d\delta^i_a)(\delta^j_b\delta^k_c + \delta^j_c\delta^k_b) \tag{3.16}$$

Upon multiplying (3.14) by $g_{iq}g_{uw}g_{rs}$ we find that equations (3.15) and (3.16) may be used to rewrite the resultant expression as follows:

$$\phi^{|a}g_{iq}g_{uw}g_{rs}B^{;iq;uw;rs,tv} = 4\sqrt{(g)}\epsilon(\phi^{|a}g^{vt} + \phi^{|t}g^{va} + \phi^{|v}g^{ta}) \tag{3.17}$$

Now given any scalar field there always locally exists a vector field X^a for which

$$\phi_{,a}X^a = 0$$

and

$$X_aX^a \neq 0$$

If we now multiply (3.17) by X_aX_t we find that

$$0 = 4\epsilon\sqrt{(g)}\phi^{|v}$$

and hence

$$\epsilon = 0$$

Using similar techniques it can be shown that if (3.11) holds then (3.12) must hold and hence B must vanish.

Consequently, $A^{ab}{}_{|b}$, as given in (3.7), will be of the form (3.2) if and only if the ten scalar functions $K_1, \ldots, K_{10}$ appearing in A^{ab} are related by the eight partial differential equations obtained by setting each of the quantities appearing in (3.9) equal to zero. Of these eight differential equations only six are independent because

$$\nu = 2\dot{\alpha} + \gamma - 2\epsilon'$$

and

$$\omega = 2\dot{\mu} - \alpha' + \rho\gamma' - 2\beta'$$

The remaining six equations imply that the ten functions appearing in A^{ab} must be related as follows if $A^{ab}{}_{|b}$ is to satisfy (3.2):

$$\left.\begin{array}{lll} K_4 = -\frac{4}{3}\dot{K}_1; & K_5 = 2K_3 - 2K_1' + 4\rho\dot{K}_3; & K_6 = -4\dot{K}_3 \\ K_2 = \frac{1}{2}F + W; & K_7 = -2F' - 4W' - \rho K_8; & \text{and} \\ K_{10} = -2F'' - 4W'' - \rho K_8' - 2\dot{K}_9 & & \end{array}\right\} \tag{3.18}$$

where K_1, K_3, K_8 and K_9 are arbitrary functions of ϕ and ρ, W is an arbitrary function of ϕ and F is given by the following indefinite integral

$$F = F(\phi\,;\rho) = \int \{K_1'(\phi;\rho) - K_3(\phi;\rho) - 2\rho\dot{K}_3(\phi;\rho)\}\,d\rho \tag{3.19}$$

Using equations (2.27) and (3.18) we obtain

Theorem 3.1. *In a space of dimension four the most general symmetric contravariant tensor density A^{ab} which is a concomitant of g_{ij}, and its first two derivatives, together with ϕ, and its first two derivatives, and is such that its covariant divergence, $A^{ab}{}_{|b}$, is of the form*

$$A^{ab}{}_{|b} = \phi^{|a} A(g_{ij}; g_{ij,h}; g_{ij,hk}; \phi; \phi_{,h}; \phi_{,hk})$$

(*where A is a scalar density*) *is given by*

$$\begin{aligned} A^{ab} = {} & \sqrt{(g)}K_1\delta^{acde}_{fhjk}g^{fb}\phi_{|c}{}^{|h}R_{de}{}^{jk} + \sqrt{(g)}(\tfrac{1}{2}F + W)\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh} \\ & + \sqrt{(g)}K_3\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}R_{de}{}^{jk} - \tfrac{4}{3}\sqrt{(g)}\dot{K}_1\delta^{acde}_{fhjk}g^{fb}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\ & + \sqrt{(g)}(2K_3 - 2K_1' + 4\rho\dot{K}_3)\delta^{acd}_{efh}g^{eb}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} \\ & - 4\sqrt{(g)}\dot{K}_3\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\ & - \sqrt{(g)}(2F' + 4W' + \rho K_8)\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e} + \sqrt{(g)}K_8\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} \\ & + \sqrt{(g)}K_9 g^{ab} - \sqrt{(g)}(2F'' + 4W'' + \rho K_8' + 2\dot{K}_9)\phi^{|a}\phi^{|b} \end{aligned} \tag{3.20}$$

where K_1, K_3, K_8 and K_9 are arbitrary scalar functions of ϕ and ρ, W is an arbitrary scalar function of ϕ, and F is given by equation (3.19).

Due to equations (3.7), (3.8) and (3.18) we find that when A^{ab} is given by equation (3.20) then

$$\begin{aligned} A^{ab}{}_{|b} = \sqrt{(g)}\phi^{|a}\{&\dot{K}_1\delta^{bcde}_{hjkm}\phi_{|b}{}^{|h}\phi_{|c}{}^{|j}R_{de}{}^{km} \\ &+(K_1'-K_3)\delta^{cde}_{fjk}\phi_{|c}{}^{|f}R_{de}{}^{jk} - 2\dot{K}_3\delta^{cde}_{fjk}\phi^{|p}\phi_{|p}{}^{|f}\phi_{|c}R_{de}{}^{jk} \\ &+4\dot{K}_3\delta^{cde}_{fjk}\phi_{|c}\phi^{|m}R_{md}{}^{jf}\phi_{|e}{}^{|k} - \tfrac{1}{8}K_1\delta^{bcde}_{hjkm}R_{bc}{}^{hj}R_{de}{}^{km} \\ &+(\tfrac{1}{2}F'+W')\delta^{cd}_{fh}R_{cd}{}^{fh} - \tfrac{1}{2}K_8\delta^{cd}_{eh}\phi_{|c}\phi^{|m}R_{md}{}^{he} \\ &-2\dot{K}_8\delta^{cd}_{eh}\phi_{|p}\phi^{|pe}\phi_{|c}\phi_{|d}{}^{|h} - \tfrac{2}{3}\ddot{K}_1\delta^{bcde}_{hjkm}\phi_{|b}{}^{|h}\phi_{|c}{}^{|j}\phi_{|d}{}^{|k}\phi_{|e}{}^{|m} \\ &+8\ddot{K}_3\delta^{cde}_{fjk}\phi^{|p}\phi_{|p}{}^{|f}\phi_{|c}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} + (4\dot{K}_3-\tfrac{4}{3}\dot{K}_1')\delta^{cde}_{hjk}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\ &+(2K_3'-2K_1''+4\rho\dot{K}_3'-K_8)\delta^{cd}_{fh}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} + K_9' \\ &-\rho(2F'''+4W'''+\rho K_8''+2\dot{K}_9') \\ &-2(2\dot{F}''+K_8'+\rho\dot{K}_8'+2\ddot{K}_9)\phi^{|c}\phi^{|d}\phi_{|cd} \\ &-(4F''+8W''+2\rho K_8'+2\dot{K}_9)\phi_{|c}{}^{|c}\} \end{aligned} \tag{3.21}$$

Remark. Using equations (3.20) and (3.21) it is easy to show that in a space of four-dimensions the most general *divergence free* symmetric tensor density of the form (3.1) is given by:

$$A^{ab} = \sqrt{(g)}c_2\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh} + \sqrt{(g)}c_1g^{ab}$$

where c_1 and c_2 are arbitrary real constants. Thus in a four-dimensional space there does not exist a *genuine* divergence free tensor density of the form (3.1); i.e., A^{ab} must be independent of ϕ and its derivatives if it is to be divergence free.

Employing techniques similar to those used to derive Theorem 3.1 we obtain

Theorem 3.2. *In spaces of dimension* 2 *and* 3 *the most general symmetric contravariant tensor density A^{ab} which is a concomitant of g_{ij}, and its first two derivatives, together with ϕ, and its first two derivatives, and is such that its covariant divergence, $A^{ab}{}_{|b}$, is of the form*

$$A^{ab}{}_{|b} = \phi^{|a}A(g_{ij};g_{ij,h};g_{ij,hk};\phi;\phi_{,h};\phi_{,hk})$$

(where A is a scalar density) is given by

$$A^{ab} = \sqrt{(g)}K_1\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e} + \sqrt{(g)}(K_1'-2\dot{K}_2)\phi^{|a}\phi^{|b} + \sqrt{(g)}K_2g^{ab} \tag{3.22}$$

in a space of dimension 2 *and*

$$\begin{aligned} A^{ab} = &\sqrt{(g)}K_3\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh} - 4\sqrt{(g)}\dot{K}_3\delta^{acd}_{efh}g^{eb}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} \\ &-\sqrt{(g)}(\rho K_4+4K_3')\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e} + \sqrt{(g)}K_4\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} \\ &+\sqrt{(g)}K_5g^{ab} - \sqrt{(g)}(\rho K_4'+4K_3''+2\dot{K}_5)\phi^{|a}\phi^{|b} \end{aligned} \tag{3.23}$$

in a space of dimension 3, *where $K_1, \ldots, K_5$ are arbitrary scalar functions of ϕ and ρ.*

4. *Lagrange Scalar Densities*

As mentioned in the introduction Lovelock (1971) has shown (in a space of arbitrary dimension) that if $\mathscr{A}^{ij}$ is a symmetric tensor density of the form

$$\mathscr{A}^{ij} = \mathscr{A}^{ij}(g_{hk}; g_{hk,c}; g_{hk,cd})$$

and is such that

$$\mathscr{A}^{ij}{}_{|j} = 0$$

then $\mathscr{A}^{ij} = E^{ij}(L)$ for some Lagrange scalar density of the following type

$$L = L(g_{hk}; g_{hk,c}; g_{hk,cd})$$

Furthermore, Lovelock has shown that an L which yields $\mathscr{A}^{ij} = E^{ij}(L)$, may be obtained by examining $g_{ij}\mathscr{A}^{ij}$.

We now desire to determine a Lagrange scalar density of the form (1.5) which yields the symmetric tensor density A^{ab} presented in equation (3.20), for a space of dimension 4. As a consequence of the above remark it would seem appropriate for us to begin our search with an examination of $L = g_{ab}A^{ab}$ and its associated Euler–Lagrange tensor, $E^{ij}(L)$. (In fact there are general grounds for expecting $E^{ij}(g_{ab}A^{ab})$ to be 'related' to A^{ij}, see Lovelock (1972)).

Using equation (3.20) we find that in a space of dimension 4, $g_{ab}A^{ab}$ is given by:

$$\begin{aligned} g_{ab}A^{ab} = \sqrt{(g)}\{&K_1\delta^{cde}_{hjk}\phi_{|c}{}^{|h}R_{de}{}^{jk} - \tfrac{4}{3}\dot{K}_1\delta^{cde}_{hjk}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\ &+ K_3\delta^{cde}_{hjk}\phi_{|c}\phi^{|h}R_{de}{}^{jk} - 4\dot{K}_3\delta^{cde}_{hjk}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\ &+ (F + 2W)\delta^{cd}_{fh}R_{cd}{}^{fh} + 2(2K_3 - 2K_1' + 4\rho\dot{K}_3)\delta^{cd}_{fh}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} \\ &- 3(2F' + 4W' + \rho K_8)\phi_{|c}{}^{|c} + 2K_8\delta^{cd}_{fh}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} \\ &+ 4K_9 - \rho(2F'' + 4W'' + \rho K_8' + 2\dot{K}_9)\} \end{aligned} \tag{4.1}$$

where F is given by (3.19).

We now wish to determine the Euler–Lagrange tensor (viz., E^{ab}) corresponding to the scalar density presented in equation (4.1). Upon a closer examination of this Lagrangian we see that in order to determine its Euler–Lagrange tensor it would suffice to know

$$E^{ab}(L_\alpha)$$

($\alpha = 1, \ldots, 6$) in a space of arbitrary dimension, where the L_α's are given by:

$$L_1 = \sqrt{(g)}M_1\phi_{|c}{}^{|c} \tag{4.2}$$

$$L_2 = \sqrt{(g)}M_2\delta^{cd}_{ef}R_{cd}{}^{ef} - 4\sqrt{(g)}\dot{M}_2\delta^{cd}_{fk}\phi_{|c}{}^{|f}\phi_{|d}{}^{|k} \tag{4.3}$$

$$L_3 = \sqrt{(g)}M_3\delta^{cd}_{ef}\phi_{|c}\phi^{|e}\phi_{|d}{}^{|f} \tag{4.4}$$

$$L_4 = \sqrt{(g)}M_4\delta^{cde}_{fjk}\phi_{|c}{}^{|f}R_{de}{}^{jk} - \tfrac{4}{3}\sqrt{(g)}\dot{M}_4\delta^{cde}_{fjk}\phi_{|c}{}^{|f}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \tag{4.5}$$

$$L_5 = \sqrt{(g)}M_5\delta^{cde}_{fjk}\phi_{|c}\phi^{|f}R_{de}{}^{jk} - 4\sqrt{(g)}\dot{M}_5\delta^{cde}_{fjk}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \tag{4.6}$$

and

$$L_6 = \sqrt{(g)}M_6 \tag{4.7}$$

where $M_1, \ldots, M_6$ are arbitrary functions of ϕ and ρ.

Using techniques similar to those employed in Rund (1964, 1966) and Horndeski & Lovelock (1972) it can be shown that if

$$L = L(g_{ij}; g_{ij,h}; g_{ij,hk}; \phi; \phi_{,h}; \phi_{,hk}) \tag{4.8}$$

then

$$E^{ab}(L) = -\pi^{ab,hk}{}_{|hk} + \pi^{ab,h}{}_{|h} - \pi^{ab} \tag{4.9}$$

and

$$E(L) = -\zeta^{hk}{}_{|hk} + \zeta^{h}{}_{|h} - \zeta \tag{4.10}$$

where

$$\left.\begin{aligned}
&\pi^{ab,hk} = \frac{\partial L}{\partial g_{ab,hk}}; \qquad \zeta^{ab} = \frac{\partial L}{\partial \phi_{,ab}}; \qquad \zeta = \frac{\partial L}{\partial \phi};\\
&\pi^{ab,h} = \tfrac{1}{2}(\zeta^{ab}\phi^{|h} - \zeta^{hb}\phi^{|a} - \zeta^{ha}\phi^{|b})\\
&\pi^{ab} = \tfrac{1}{3}R_{k}{}^{b}{}_{mh}\pi^{hk,am} - R_{k}{}^{a}{}_{mh}\pi^{hk,bm}\\
&\qquad -\tfrac{1}{2}\phi^{|a}\zeta^{|b} - \zeta^{bs}\phi_{|s}{}^{|a} + \tfrac{1}{2}g^{ab}L\\
&\text{and}\\
&\zeta^{a} = \frac{\partial L}{\partial \phi_{,a}} + \Gamma_{r}{}^{a}{}_{s}\zeta^{rs}
\end{aligned}\right\} \tag{4.11}$$

Employing equations (3.15), (3.16), (4.9) and (4.11) we find, after a lengthy, but straightforward, calculation, that in a space of any dimension (for the details see Horndeski (1973))

$$\begin{aligned}
E^{ab}(L_1) = {} & \sqrt{(g)}\rho\dot{M}_1\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e} - \sqrt{(g)}\dot{M}_1\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h}\\
& + \sqrt{(g)}M_1'(\tfrac{1}{2}g^{ab}\rho - \phi^{|a}\phi^{|b})
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
E^{ab}(L_2) = {} & -\sqrt{(g)}\dot{M}_2\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}R_{de}{}^{jk} + \sqrt{(g)}(\rho\dot{M}_2 - \tfrac{1}{2}M_2)\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh}\\
& -2\sqrt{(g)}(2\rho\ddot{M}_2 + \dot{M}_2)\delta^{acd}_{efh}g^{eb}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h}\\
& +4\sqrt{(g)}\ddot{M}_2\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} + 2\sqrt{(g)}(M_2' + 2\rho\dot{M}_2')\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e}\\
& -8\sqrt{(g)}\dot{M}_2'\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} + 2\sqrt{(g)}\rho M_2''g^{ab} - 2\sqrt{(g)}M_2''\phi^{|a}\phi^{|b}
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
E^{ab}(L_3) = {} & \sqrt{(g)}(\rho^2\dot{M}_3 + \tfrac{3}{2}\rho M_3)\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e}\\
& -\sqrt{(g)}(\rho\dot{M}_3 + \tfrac{3}{2}M_3)\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} + \tfrac{1}{2}\sqrt{(g)}M_3'\rho^2g^{ab}\\
& -\tfrac{1}{2}\sqrt{(g)}\rho M_3'\phi^{|a}\phi^{|b}
\end{aligned} \tag{4.14}$$

$$E^{ab}(L_4) = -\sqrt{(g)}\dot{M}_4\delta^{acdef}_{hjkpq}g^{hb}\phi_{|c}\phi^{|j}\phi_{|d}{}^{|k}R_{ef}{}^{pq}$$
$$+ \sqrt{(g)}\dot{M}_4\rho\,\delta^{adef}_{hkpq}g^{hb}\phi_{|d}{}^{|k}R_{ef}{}^{pq} - \tfrac{1}{2}\sqrt{(g)}\rho M_4'\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh}$$
$$+ \sqrt{(g)}M_4'\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}R_{de}{}^{jk}$$
$$+\tfrac{4}{3}\sqrt{(g)}\ddot{M}_4\delta^{acdef}_{hjkpq}g^{hb}\phi_{|c}\phi^{|j}\phi_{|d}{}^{|k}\phi_{|e}{}^{|p}\phi_{|f}{}^{|q}$$
$$-\tfrac{4}{3}\sqrt{(g)}(\rho\ddot{M}_4 + \dot{M}_4)\delta^{acde}_{fhjk}g^{fb}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}$$
$$+ \sqrt{(g)}(2M_4' + 2\rho\dot{M}_4')\delta^{ade}_{fjk}g^{fb}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}$$
$$-4\sqrt{(g)}\dot{M}_4'\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} + 2\sqrt{(g)}M_4''\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} \tag{4.15}$$

$$E^{ab}(L_5) = \sqrt{(g)}(\dot{M}_5\rho^2 + \tfrac{1}{2}\rho M_5)\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh}$$
$$-\sqrt{(g)}(\rho\dot{M}_5 + M_5)\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}R_{de}{}^{jk}$$
$$-\sqrt{(g)}(2M_5 + 10\rho\dot{M}_5 + 4\rho^2\ddot{M}_5)\delta^{acd}_{efh}g^{eb}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h}$$
$$+4\sqrt{(g)}(2\dot{M}_5 + \rho\ddot{M}_5)\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}$$
$$-\sqrt{(g)}(2M_5' + 4\rho\dot{M}_5')\delta^{ace}_{jfr}g^{jb}\phi_{|c}\phi^{|f}\phi_{|e}{}^{|r} \tag{4.16}$$

and

$$E^{ab}(L_6) = \sqrt{(g)}\dot{M}_6\phi^{|a}\phi^{|b} - \tfrac{1}{2}\sqrt{(g)}M_6g^{ab} \tag{4.17}$$

Making use of equations (4.1) and (4.12)-(4.17) we find that in a space of four-dimensions

$$E^{ab}(g_{cd}A^{cd}) = \sqrt{(g)}\rho\dot{K}_1\delta^{adef}_{hkpq}g^{hb}\phi_{|d}{}^{|k}R_{ef}{}^{pq}$$
$$+ \sqrt{(g)}\rho\dot{K}_3\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}R_{de}{}^{jk} + \sqrt{(g)}(\tfrac{1}{2}J - W)\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh}$$
$$-\tfrac{4}{3}\sqrt{(g)}\frac{\partial}{\partial\rho}(\rho\dot{K}_1)\delta^{acde}_{fhjk}g^{fb}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}$$
$$-4\sqrt{(g)}\frac{\partial}{\partial\rho}(\rho\dot{K}_3)\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}$$
$$+ \sqrt{(g)}(-2\rho\dot{K}_1' + 6\rho\dot{K}_3 + 4\rho^2\ddot{K}_3)\delta^{ade}_{fjk}g^{fb}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}$$
$$+ \sqrt{(g)}\rho\dot{K}_8\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} + \sqrt{(g)}(-2J' + 4W' - \rho^2\dot{K}_8)$$
$$\times\,\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e} + \sqrt{(g)}(\rho\dot{K}_9 - 2K_9)g^{ab}$$
$$+ \sqrt{(g)}(-2J'' + 4W'' + 2\dot{K}_9 - 2\rho\ddot{K}_9 - \rho^2\dot{K}_8')\phi^{|a}\phi^{|b} \tag{4.18}$$

where $J = J(\phi;\rho)$ is defined by the indefinite integral

$$J = \int\left\{\frac{\partial}{\partial\phi}(\rho\dot{K}_1(\phi;\rho)) - \rho\dot{K}_3(\phi;\rho) - 2\rho\frac{\partial}{\partial\rho}(\rho\dot{K}_3(\phi;\rho))\right\}d\rho \tag{4.19}$$

and integration by parts has been used to show that

$$-F + \rho\dot{F} = J \tag{4.20}$$

Upon comparing equation (4.18) with (3.20) we readily deduce that in a space of four-dimensions the symmetric tensor density presented in (3.20) is the Euler–Lagrange tensor corresponding to

$$\begin{aligned}\mathscr{L} = {} & \sqrt{(g)}\,\mathscr{K}_1 \delta^{cde}_{hjk} \phi_{|c}{}^{|h} R_{de}{}^{jk} - \tfrac{4}{3}\sqrt{(g)}\,\dot{\mathscr{K}}_1 \delta^{cde}_{hjk} \phi_{|c}{}^{|h} \phi_{|d}{}^{|j} \phi_{|e}{}^{|k} \\ & + \sqrt{(g)}\,\mathscr{K}_3 \delta^{cde}_{hjk} \phi_{|c} \phi^{|h} R_{de}{}^{jk} - 4\sqrt{(g)}\,\dot{\mathscr{K}}_3 \delta^{cde}_{hjk} \phi_{|c} \phi^{|h} \phi_{|d}{}^{|j} \phi_{|e}{}^{|k} \\ & + \sqrt{(g)}(\mathscr{F} + 2\mathscr{W})\delta^{cd}_{fh} R_{cd}{}^{fh} + 2\sqrt{(g)}(2\mathscr{K}_3 - 2\mathscr{K}'_1 + 4\rho\dot{\mathscr{K}}_3)\delta^{cd}_{fh}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} \\ & -3\sqrt{(g)}(2\mathscr{F}' + 4\mathscr{W}' + \rho\mathscr{K}_8)\phi_{|c}{}^{|c} + 2\sqrt{(g)}\,\mathscr{K}_8 \delta^{cd}_{fh}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} \\ & + \sqrt{(g)}\{4\mathscr{K}_9 - \rho(2\mathscr{F}'' + 4\mathscr{W}'' + \rho\mathscr{K}'_8 + 2\dot{\mathscr{K}}_9)\}\end{aligned} \tag{4.21}$$

where

$$\left.\begin{aligned} \mathscr{K}_1 &= \int \frac{1}{\rho} K_1(\phi;\rho)\,d\rho; & \mathscr{K}_3 &= \int \frac{1}{\rho} K_3(\phi;\rho)\,d\rho \\ \mathscr{K}_8 &= \int \frac{1}{\rho} K_8(\phi;\rho)\,d\rho; & \mathscr{K}_9 &= \rho^2 \int \frac{1}{\rho^3} K_9(\phi;\rho)\,d\rho \\ & & \mathscr{W} &= -W \end{aligned}\right\} \tag{4.22a}$$

and

$$\mathscr{F} = \int \{\mathscr{K}'_1(\phi;\rho) - \mathscr{K}_3(\phi;\rho) - 2\rho\dot{\mathscr{K}}_3(\phi;\rho)\}\,d\rho \tag{4.22b}$$

To recapitulate the above work we have

Theorem 4.1. *In a space of four-dimensions any symmetric contravariant tensor density of rank* 2 *which is a concomitant of a pseudo-Riemannian metric tensor (with components g_{ij}), and its first two derivatives, together with a scalar field ϕ, and its first two derivatives, and furthermore is such that its components, A^{ab}, satisfy*

$$A^{ab}{}_{|b} = \phi^{|a} A(g_{ij}; g_{ij,h}; g_{ij,hk}; \phi; \phi_{,h}; \phi_{,hk})$$

is the Euler–Lagrange tensor corresponding to a suitably chosen Lagrange scalar density of the form presented in equation (4.21).

Remark. The Lagrangian which yields the tensor density mentioned in Theorem 4.1 is unique only up to the addition of scalar densities of the form (1.5) which yield identically vanishing Euler–Lagrange tensors upon varying the g_{ij}'s.

As an immediate consequence of Theorem 4.1 we obtain

Theorem 4.2. *In spaces of four-dimensions the most general Euler–Lagrange equations which are at most of second-order in the derivatives of both g_{ij} and ϕ, and which are derivable from a Lagrange scalar density of the form* (1.5) *can be obtained from the Lagrangian presented in equation* (4.21) *and are given by*

$$A^{ab} = 0$$

and

$$\frac{2\phi_{,a}A^{ab}{}_{|b}}{\rho} = 0$$

where A^{ab} *and* $A^{ab}{}_{|b}$ *are given by equations* (3.20) *and* (3.21) *respectively.*

The above theorem is false in spaces of dimension greater than 4. For if K is an arbitrary scalar function of ϕ and ρ

$$\begin{aligned} L = {} & \sqrt{(g)}K\,\delta^{abcde}_{fhijk}\,\phi_{|a}{}^{|f}R_{bc}{}^{hi}R_{de}{}^{jk} \\ & -\tfrac{8}{3}\sqrt{(g)}\dot{K}\,\delta^{abcde}_{fhijk}\,\phi_{|a}{}^{|f}\phi_{|b}{}^{|h}\phi_{|c}{}^{|i}R_{de}{}^{jk} \\ & +\tfrac{16}{15}\sqrt{(g)}\ddot{K}\,\delta^{abcde}_{fhijk}\,\phi_{|a}{}^{|f}\phi_{|b}{}^{|h}\phi_{|c}{}^{|i}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} + \mathscr{L} \end{aligned}$$

(where $\mathscr{L}$ is given by equation (4.21)) provides us with a counter-example.

In the introduction it was pointed out that the most general second-order Euler–Lagrange tensors derivable from a Lagrange scalar density of the form (1.1) (in a space of four-dimensions) are given by equations (1.2) and (1.3), and may be obtained from the Lagrangian (1.4). It should be noted that the primary difference between the Euler–Lagrange tensors (1.2), (1.3) and the Euler–Lagrange tensors derivable from $\mathscr{L}$ (viz., (3.20) and $2\sqrt{(g)}$ times the term appearing within curly brackets on the right-hand side of (3.21)) lie in the following two areas:

(i) the latter Euler–Lagrange tensors involve a total of five arbitrary functions (four are concomitants of ϕ and ρ and one is a function of ϕ), whereas the former Euler–Lagrange tensors involve only four arbitrary functions (one is a concomitant of ϕ and ρ, and three are functions of ϕ);

(ii) the latter Euler–Lagrange tensors are much more non-linear in the second derivatives of ϕ and g_{ij} than are the former.

If one were now to demand that the field equations derivable from (4.21) be quasi-linear in the second derivatives of both g_{ij} and ϕ (in the sense that the coefficients of $g_{ij,hk}$ and $\phi_{,hk}$ are at most functions of g_{ij} and ϕ) then it is easily shown that a Lagrangian which yields these quasi-linear field equations is given by

$$\tilde{\mathscr{L}} = \sqrt{(g)}(f_1 R + f_2\rho + f_3) \tag{4.23}$$

(where f_1, f_2 and f_3 are arbitrary functions of ϕ) and the field equations are

$$\begin{aligned} \sqrt{(g)}\{f_1 G^{ab} - f_1'\phi^{|ab} + (f_2 - f_1'')\phi^{|a}\phi^{|b} \\ + g^{ab}[(f_1'' - \tfrac{1}{2}f_2)\rho + f_1' g^{cd}\phi_{|cd} - \tfrac{1}{2}f_3]\} = 0 \end{aligned} \tag{4.24}$$

and

$$-\sqrt{(g)}\{f_1'R - f_2'\rho - 2f_2 g^{cd}\phi_{|cd} + f_3'\} = 0 \tag{4.25}$$

Equations (4.24) *and* (4.25) *embody the most general second-order quasi-linear Euler–Lagrange equations which can be derived from a Lagrange scalar density of the form* (1.5) *in a space of four-dimensions.*

The Lagrange scalar density (4.23) is precisely the Lagrangian discussed in detail by Bergmann (1968).

Remark. It is not difficult to prove that in any space of dimension ⩾4 the most general second-order Euler–Lagrange equations derivable from a scalar density of the form (1.5) which are in addition quasi-linear in the second derivatives of both g_{ij} and ϕ are given by equations (4.24) and (4.25) and may be obtained from the Bergmann Lagrangian (4.23).

In conclusion I would like to point out that the tensor densities presented in equations (3.22) and (3.23) are the Euler–Lagrange tensors corresponding to

$$\mathscr{L}_2 = \sqrt{(g)}\,\mathscr{M}_1\phi_{|c}{}^{|c} + \sqrt{(g)}(\mathscr{M}'_1 - 2\dot{\mathscr{M}}_2)\rho + 2\sqrt{(g)}\,\mathscr{M}_2 \qquad (4.26)$$

(in a space of two-dimensions) and

$$\begin{aligned}\mathscr{L}_3 = {} & \sqrt{(g)}\,\mathscr{M}_3\delta^{cd}_{fh}R_{cd}{}^{fh} - 4\sqrt{(g)}\dot{\mathscr{M}}_3\delta^{cd}_{fh}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} \\ & + -2\sqrt{(g)}(\rho\,\mathscr{M}_4 + 4\mathscr{M}'_3)\phi_{|c}{}^{|c} + \sqrt{(g)}\,\mathscr{M}_4\delta^{cd}_{fh}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} \\ & + \sqrt{(g)}[3\mathscr{M}_5 - (\rho\mathscr{M}'_4 + 4\mathscr{M}''_3 + 2\dot{\mathscr{M}}_5)\rho] \qquad (4.27)\end{aligned}$$

(in a space of three-dimensions) respectively, where

$$\mathscr{M}_1 = \int \frac{1}{\rho}K_1(\phi;\rho)\,d\rho; \qquad \mathscr{M}_2 = \rho\int \frac{1}{\rho^2}K_2(\phi;\rho)\,d\rho;$$

$$\mathscr{M}_3 = \sqrt{(\rho)}\int \frac{1}{\rho^{3/2}}K_3(\phi;\rho)\,d\rho; \qquad \mathscr{M}_4 = \frac{1}{\sqrt{\rho}}\int \frac{1}{\sqrt{\rho}}K_4(\phi;\rho)\,d\rho;$$

and

$$\mathscr{M}_5 = \rho^{3/2}\int \frac{1}{\rho^{5/2}}K_5(\phi;\rho)\,d\rho$$

with $K_1, \ldots, K_5$ being the scalar functions which appear in equations (3.22) and (3.23).

Consequently, in spaces of dimension two and three the most general Euler–Lagrange equations which are at most of second order in the derivatives of both g_{ij} and ϕ, and furthermore are derivable from a Lagrange scalar density of the form (1.5), may be obtained from the Lagrangians (4.26) and (4.27) respectively.

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